MODEL CATEGORY STRUCTURES ON CHAIN COMPLEXES OF SHEAVES

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ABSTRACT. In this paper, we try to determine when the derived category of an abelian category is the homotopy category of a model structure on the category of chain complexes. We prove that this is always the case when the abelian category is a Grothendieck category, as has also been done by Morel. But this model structure is not very useful for defining derived tensor products. We therefore consider another method for constructing a model structure, and apply it to the category of sheaves on a well-behaved ringed space. The resulting flat model structure is compatible with the tensor product and all homomorphisms of ringed spaces.

INTRODUCTION

It very often happens in mathematics that one has a category $\mathcal{C}$ and a collection of maps $\mathcal{W}$ in $\mathcal{C}$ that one would like to consider as isomorphisms. In this situation, one can formally invert the maps in $\mathcal{W}$, but the resulting localization $\operatorname{Ho}\mathcal{C}$ of $\mathcal{C}$ may not be a category in general, because $\operatorname{Ho}\mathcal{C}(X,Y)$ may not be a set. Furthermore, it is hard to get a handle on maps in $\operatorname{Ho}\mathcal{C}$ from X to Y. Model categories were invented by Quillen [Qui67] to get around these problems. In general, it is hard to prove a given category $\mathcal{C}$ is a model category, but, having done so, many structural results about $\operatorname{Ho}\mathcal{C}$ follow easily; for example, $\operatorname{Ho}\mathcal{C}$ is then canonically enriched over the homotopy category of simplicial sets. And of course one can then use the considerable body of results about model categories to investigate $\mathcal{C}$.

An obvious example of a situation where one wants to invert some maps is the construction of the derived category of an abelian category $\mathcal{A}$. Recall that this is the localization of the category $\operatorname{Ch}(\mathcal{A})$ of (unbounded) chain complexes by maps which induce homology isomorphisms. The category of nonnegatively graded chain complexes of R-modules was one of Quillen's first examples of a model category. Nevertheless, the first published proof that the category of unbounded chain complexes of R-modules is a model category appears to be in [Hov98].

We begin the paper by establishing a model structure on $\operatorname{Ch}(\mathcal{A})$ whose homotopy category is the derived category, when $\mathcal{A}$ is a a Grothendieck category. In particular, $\mathcal{A}$ could be the category of sheaves on a ringed space, or the category of quasi-coherent sheaves on a quasi-compact and quasi-separared scheme. The author has heard that this has been done by F. Morel as well, but does not know any details. The injective model structure is natural for exact functors of abelian categories, but not for left adjoints which are only right exact. Also, if $\mathcal{A}$ is closed symmetric monoidal, the injective model structure will not be compatible with the tensor product, making it of no use for defining the derived tensor product.

We therefore discuss a different method for constructing a model structure on $\mathrm{Ch}(\mathcal{A})$. This method enables us to define a different model structure on $\mathrm{Ch}(\mathcal{A})$ in case $\mathcal{A}$ has a set of generators of finite projective dimension. In particular, we apply it when $\mathcal{A}$ is the category of quasi-coherent sheaves on a nice enough scheme, using the locally free sheaves as the generators. Though the resulting locally free model structure is still not compatible with the tensor product, it does give us some information about the resulting derived category that does not seem accessible from the injective model structure.

But this method works better when $\mathcal{A}$ is the category of sheaves on a ringed space $(S, \mathcal{O})$ satisfying a hypothesis related to finite cohomological dimension. In this case, we construct a flat model structure on $\mathrm{Ch}(\mathcal{A})$ that is compatible with the tensor product. We then get model categories of differential graded $\mathcal{O}$-algebras and of differential graded modules over a given differential graded $\mathcal{O}$-algebra. The flat model structure is also natural for arbitrary maps of ringed spaces.

To understand this paper, the reader needs to know some basic facts about model categories, Grothendieck categories, and sheaves. A good introduction to model categories is [DS95]. The book [Hov98] is a more in-depth study, but still starting from scratch. All the terms we need are defined in [Hov98]; we will give specific references as needed. For Grothendieck categories, [Ste75] is sufficient. For sheaves and schemes, we try to refer mostly to [Har77], but we also need more advanced results occasionally.

The author would like to thank Matthew Ando and Amnon Neeman for helpful discussions about sheaves, and Dan Christensen for his many useful suggestions.

1. Grothendieck categories

In this section, we develop the basic structural properties of Grothendieck categories that we need. In particular, we show that every object in a Grothendieck category is small and that the fundamental lemma of homological algebra holds in a Grothendieck category.

Recall that a cocomplete abelian category is called an AB5 category if directed colimits, or, equivalently, filtered colimits, are exact. A Grothendieck category is an AB5 category that has a generator. Recall that U is a generator of $\mathcal{A}$ if the functor $\mathcal{A}(U, -)$ is faithful. For example, the category of sheaves on a ringed space, or a ringed topos, is a Grothendieck category. The Popescu-Gabriel theorem, a proof of which can be found in [Ste75, Section X.4], asserts that every Grothendieck category $\mathcal{A}$ is equivalent to the full subcategory of $\mathcal{T}$-local objects in a module category R-Mod, for some hereditary torsion theory $\mathcal{T}$ and some ring R. The ring R can be taken to be $\mathrm{Hom}(U, U)$ for a generator U, but since the generator U is not canonically attached to $\mathcal{A}$, neither is the ring R.

One of the most basic tools used to establish the existence of a model category structure is the small object argument [Hov98, Section 2.1]. In order to use the small object argument, we need to know that the objects in a Grothendieck category are small; *i.e.* that maps out of an object commute with long enough colimits. In order to make this precise, we need some definitions.

Definition 1.1.   1. Given a limit ordinal λ, the *cofinality* of λ, cofin λ, is the smallest cardinal κ such that there exists a subset T of λ with $|T| = \kappa$ and $\sup T = \lambda$.

2. Given an object A in a cocomplete category $\mathcal{C}$ and a cardinal κ, we say that A is κ-*small* if, for every ordinal λ with cofin $\lambda > \kappa$ and every colimit-preserving functor $X\colon \lambda \to \mathcal{C}$, the natural map $\mathrm{colim}_{i<\lambda}\, \mathcal{C}(A, X_i) \to \mathcal{C}(A, \mathrm{colim}_{i<\lambda} X_i)$ is an isomorphism.

3. An object A in a cocomplete category $\mathcal{C}$ is called *small* if it is κ-small for some cardinal κ.

Proposition 1.2. *Every object in a Grothendieck category is small.*

We do not know if this proposition holds more generally, `e.g.`, in any AB5 category, but it seems unlikely.

Proof. We may as well assume that our Grothendieck category $\mathcal{A}$ is the localization of R-Mod with respect to a hereditary torsion theory $\mathcal{T}$, for some ring R. Let κ be the larger of ∞ and the cardinality of R, let λ be an ordinal with cofin $\lambda > \kappa$, and let $X\colon \lambda \to \mathcal{A}$ be a colimit-preserving functor. We will first show that $\mathrm{colim}\, X_i$, calculated in R-Mod, is still $\mathcal{T}$-local, so is also the colimit in $\mathcal{A}$. This proof will depend on the fact that both $R/\mathfrak{a}$ and $\mathfrak{a}$ are κ-small in R-Mod [Hov98, Example 2.1.6], for all (left) ideals $\mathfrak{a}$ of R.

To see this, first note that $\mathrm{colim}\, X_i$ is torsion-free. Indeed, $\mathcal{T}$ is generated by cyclic modules $R/\mathfrak{a}$, so it suffices to show that $R\text{-Mod}(R/\mathfrak{a}, \mathrm{colim}\, X_i) = 0$. But we have chosen κ so that

$$R\text{-Mod}(R/\mathfrak{a}, \mathrm{colim}\, X_i) \cong \mathrm{colim}\, R\text{-Mod}(R/\mathfrak{a}, X_i) = 0,$$

since each X_i is torsion-free. Hence $\mathrm{colim}\, X_i$ is torsion-free.

It follows that the localization of $\mathrm{colim}\, X_i$ is

$$\mathrm{colim}_{\mathfrak{a}} \mathrm{Hom}(\mathfrak{a}, \mathrm{colim}_i X_i),$$

where the colimit is taken over ideals $\mathfrak{a}$ such that $R/\mathfrak{a}$ is torsion, as in [Ste75, Section IX.1]. But then we have

$$\mathrm{colim}_{\mathfrak{a}} \mathrm{Hom}(\mathfrak{a}, \mathrm{colim}_i X_i) \cong \mathrm{colim}_{\mathfrak{a}} \mathrm{colim}_i \mathrm{Hom}(\mathfrak{a}, X_i)$$
$$\cong \mathrm{colim}_i \mathrm{colim}_{\mathfrak{a}} \mathrm{Hom}(\mathfrak{a}, X_i) = \mathrm{colim}_i X_i.$$

Thus $\mathrm{colim}\, X_i$ is already local.

Now suppose M is an arbitrary local module. There is a cardinal κ' such that M is κ'-small as an R-module, and we can choose $\kappa' \geq \kappa$. It is then immediate from the argument above that M is κ'-small in $\mathcal{A}$. $\qquad\square$

We will be most interested in the category of unbounded chain complexes $\mathrm{Ch}(\mathcal{A})$ on a Grothendieck category $\mathcal{A}$. This is again a Grothendieck category; since colimits are taken dimensionwise, filtered colimits are obviously exact, and if U is a generator of $\mathcal{A}$, then the disks $D^n U$ are generators of $\mathrm{Ch}(\mathcal{A})$. Recall that $D^n U$ is the complex which is U in degrees n and $n-1$ and 0 elsewhere, with the interesting differential being the identity map. To see that the disks generate $\mathrm{Ch}(\mathcal{A})$, use the adjunction relation $\mathrm{Ch}(\mathcal{A})(D^n U, X) \cong \mathcal{A}(U, X_n)$. In particular, every object of $\mathrm{Ch}(\mathcal{A})$ is small.

We recall the fundamental lemma of homological algebra, which holds in any abelian category [Mit65, Section VI.8].

Lemma 1.3. *Suppose $\mathcal{A}$ is an abelian category, and*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence in $\mathrm{Ch}(\mathcal{A})$. *Then there is a natural long exact sequence*

$$\ldots \to H_{n+1}C \xrightarrow{\partial_*} H_n A \xrightarrow{f_*} H_n B \xrightarrow{g_*} H_n C \xrightarrow{\partial_*} H_{n-1}A \to \ldots$$

in homology.

Recall that a map f in $\mathrm{Ch}(\mathcal{A})$ is called a *quasi-isomorphism* if $H_* f$ is an isomorphism. The following corollary is immediate.

Corollary 1.4. *Suppose* $\mathcal{A}$ *is an abelian category, and*

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\
C & \xrightarrow[\ g\ ]{} & D
\end{array}
$$

is a commutative square in $\mathrm{Ch}(\mathcal{A})$.

(a) *If the square above is a pushout square and f is an injective quasi-isomorphism, so is g.*

(b) *If the square above is a pushout square, r is injective and f is a quasi-isomorphism, then g is a quasi-isomorphism.*

(c) *If the square above is a pullback square and g is a surjective quasi-isomorphism, so is f.*

(d) *If the square above is a pullback square, s is surjective, and g is a quasi-isomorphism, then f is a quasi-isomorphism.*

It is also useful to know that homology commutes with colimits.

Lemma 1.5. *Suppose* $\mathcal{A}$ *is an AB5 category,* $\mathcal{I}$ *is a small filtered category, and* $F\colon \mathcal{I} \to \mathrm{Ch}(\mathcal{A})$ *is a functor. Then there is a natural isomorphism* $\mathrm{colim}\, H_n F \to H_n\, \mathrm{colim}\, F$.

Proof. We have an exact sequence

$$0 \to Z_n F(i) \to F(i)_n \xrightarrow{d} F(i)_{n-1} \to B_n F(i) \to 0$$

for each object i of $\mathcal{I}$. Since filtered colimits are exact, we find that $Z_n \mathrm{colim}_i F(i) \cong \mathrm{colim}_i Z_n F(i)$ and smiilarly for B_n. Applying colimits to the short exact sequences

$$0 \to B_n F(i) \to Z_n F(i) \to H_n F(i)$$

completes the proof. $\qquad\square$

In particular, using transfinite induction, we obtain the following proposition.

Proposition 1.6. *Suppose* $\mathcal{A}$ *is an AB5 category,* λ *is an ordinal, and* $X\colon \lambda \to \mathrm{Ch}(\mathcal{A})$ *is a colimit-preserving functor such that, for all $\alpha < \lambda$, the map $X_\alpha \to X_{\alpha+1}$ is a quasi-isomorphism. Then the map $X_0 \to \mathrm{colim}_{\alpha<\lambda} X_\alpha$ is a quasi-isomorphism.*

In the theory of model categories, one frequently has a set of maps J and wants to consider the class J-inj of maps that look like fibrations to J and the class J-cof of maps that look like cofibrations to J-inj. These classes are defined by lifting properties [Hov98, Section 2.1].

Corollary 1.7. *Let* $\mathcal{A}$ *be a Grothendieck category. Suppose J is a set of injective quasi-isomorphisms in* $\mathrm{Ch}(\mathcal{A})$. *Then J-cof consists of injective quasi-isomorphisms.*

Proof. By the small object argument [Hov98, Theorem 2.1.14 and Corollary 2.1.15], every element of J-cof is a retract of a transfinite composition of pushouts of elements of J. Injections are closed under retracts and pushouts in any abelian category; the AB5 condition guarantees that they are also closed under transfinite compositions. Part (a) of Corollary 1.4 then shows that pushouts of maps of J are quasi-isomorphisms. Proposition 1.6 shows that transfinite compositions of quasi-isomorphisms are quasi-isomorphisms. It is clear that retracts of injective quasi-isomorphisms are quasi-isomorphisms. $\square$

Note that this corollary holds in any AB5 category as long as the domains and codomains of the maps of J are small, so that the small object argument applies.

2. The injective model structure

In this section, we construct the injective model structure on $\mathrm{Ch}(\mathcal{A})$ when $\mathcal{A}$ is a Grothendieck category.

Definition 2.1. Define a map $p\colon X \to Y$ in $\mathrm{Ch}(\mathcal{A})$ to be an *injective fibration* if it has the right lifting property with respect to all injective weak equivalences in $\mathrm{Ch}(\mathcal{A})$.

Note that, by definition, a complex X is injectively fibrant if and only if X is *DG*-injective in the sense of [AFH97, Section 7]. The arguments in that paper then show that X is *DG*-injective if and only if each X_n is injective in $\mathcal{A}$ and X is K-injective in the sense of Spaltenstein [Spa88]. We will see in Proposition 2.12 that an injective fibration is just a dimensionwise split surjection with *DG*-injective kernel.

Then the object of this section is to prove the following theorem.

Theorem 2.2. *Suppose $\mathcal{A}$ is a Grothendieck category. Then the derived category of $\mathcal{A}$ is the homotopy category of a cofibrantly generated proper model structure on $\mathrm{Ch}(\mathcal{A})$ where the cofibrations are the injections, the fibrations are the injective fibrations, and the weak equivalences are the quasi-isomorphisms.*

We call this model structure on $\mathrm{Ch}(\mathcal{A})$ the *injective structure*, or the *injective model structure*.

Corollary 2.3. (a) *Suppose $(S, \mathcal{O})$ is a ringed space (or a ringed topos). Then the injective structure on the category $\mathrm{Ch}(\mathcal{O}\text{-}Mod)$ of unbounded complexes of sheaves of $\mathcal{O}$-modules is a model structure, whose homotopy category is the derived category.*

(b) *Suppose S is a quasi-compact, quasi-separated scheme. Then the injective structure on the category $\mathrm{Ch}_{\mathrm{QCo}}(S)$ of unbounded complexes of quasi-coherent sheaves of $\mathcal{O}_S$-modules is a model structure, whose homotopy category is the derived category of quasi-coherent sheaves.*

Proof. It is well-known that the category of sheaves on a ringed space is a Grothendieck category [Gro57, Proposition 3.1.1]. The category of quasi-coherent sheaves on ringed space is an abelian subcategory of all sheaves, closed under colimits. So colimits are exact. It remains to show that the category of quasi-coherent sheaves on a quasi-compact, quasi-separated scheme has a generator. This is a corollary of Deligne's result in [Har66, Appendix, Prop. 2], which asserts that every quasi-coherent sheaf is the colimit of finitely presented sheaves. Since there is only a set of finitely presented sheaves, the direct sum of all of them will serve as a generator. $\square$

Theorem 2.2 is of course a generalization of the corresponding theorem about complexes of modules [Hov98, Theorem 2.3.13], and will be proved in a similar way. We need sets I and J of generating cofibrations and generating trivial cofibrations. We will find these sets by just taking all injections (resp. all injective weak equivalences) whose cardinality is not too large. Then we will show that I-cof is the class of injections and J-cof is the class of injective weak equivalences. We must also show that every map in I-inj is an injective fibration and a weak equivalence. The recognition theorem [Hov98, Theorem 2.1.19] will then prove that we do get a model category. Properness is automatic from Corollary 1.4.

To carry out this plan, we need a notion of cardinality. This notion will depend on a way of representing our Grothendieck category $\mathcal{A}$ as a localization of a category of modules, but the resulting model structure will be independent of this choice. So throughout the rest of this section, we will assume that $\mathcal{A}$ is the localization of R-Mod, for some ring R, with respect to a hereditary torsion theory. In particular, we will think of objects of $\mathcal{A}$ as being R-modules.

Definition 2.4. Suppose $\mathcal{A}$ is a Grothendieck category. Define the *cardinality* of X, $|X|$, to be the cardinality of X as an R-module. Given a chain complex $X \in \mathrm{Ch}(\mathcal{A})$, we define $|X|$ to be the cardinality of the disjoint union of the X_n. Define γ to be the supremum of ∞ and $2^{|R|}$. Then define I to be a set containing one element of each isomorphism class of injections $A \to B$ in $\mathrm{Ch}(\mathcal{A})$ with $|B| \leq \gamma$. Define J to be the set of all quasi-isomorphisms in I.

The reason for choosing γ as we have done is the following lemma.

Lemma 2.5. *Suppose R is a ring, $\mathcal{T}$ is a torsion theory on R-Mod, and*

$$L \colon R\text{-}Mod \to R\text{-}Mod$$

is the corresponding localization functor. If M is an R-module with $|M| \leq \gamma$, then $|LM| \leq \gamma$.

Proof. We may as well assume M is torsion-free, since killing the torsion only decreases the cardinality of M, without changing LM. Then $LM = \mathrm{colim}\,\mathrm{Hom}(\mathfrak{a}, M)$, where the colimit is taken over ideals $\mathfrak{a}$ such that $R/\mathfrak{a}$ lies in $\mathcal{T}$. Thus

$$|LM| \leq 2^{|R|}|M|^{|R|}.$$

In case $|R|$ is finite, $\gamma = \infty$, and one can easily see from this equation that $|LM|$ is countable when $|M|$ is so. In case R is infinite, we have

$$|LM| \leq 2^{|R|}(2^{|R|})^{|R|} = 2^{|R|^2} = 2^{|R|} = \gamma,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As a simple case of this lemma, we have the following corollary.

Corollary 2.6. *Suppose $\mathcal{A}$ is a Grothendieck category, $X \in \mathrm{Ch}(\mathcal{A})$, and $x \in X_n$ for some n. Then the smallest subcomplex Y of X in $\mathrm{Ch}(\mathcal{A})$ containing x has $|Y| \leq \gamma$.*

Proof. The smallest subcomplex Y' of R-modules containing x is simply $R/\mathrm{ann}(x)$ in degree n and $R/\mathrm{ann}(dx)$ in degree $n-1$, so $|Y'| \leq \gamma$. Since localization is exact, the localization Y of Y' will be a subcomplex in $\mathrm{Ch}(\mathcal{A})$ containing x. Lemma 2.5 guarantees that $|Y| \leq \gamma$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We also need the following standard lemma. Given a class of maps K, K-proj is the class of maps which look projective to K; that is, they have the left lifting property with respect to K. See [Hov98, Section 2.1] for the precise definition.

Lemma 2.7. *Let $\mathcal{A}$ denote an abelian category with enough injectives. Let K denote the class of surjections in $\mathrm{Ch}(\mathcal{A})$ whose kernel is an injective object of $\mathrm{Ch}(\mathcal{A})$. Then K-proj is the class of injections. Furthermore, $(K$-proj$)$-inj $= K$.*

Proof. We first show that any map in K-proj is injective. Recall the disk functor $D^n\colon \mathcal{A} \to \mathrm{Ch}(\mathcal{A})$ that takes an object X to the complex which is X in degrees n and $n-1$, and 0 elsewhere. The functor D^n is right adjoint to the exact functor $X \mapsto X_{n-1}$. Thus $D^n(X)$ is injective whenever X is injective in $\mathcal{A}$. In particular, suppose $i\colon A \to B$ is a map of complexes with kernel C. Fix n, and embed C_n into an injective object M. This embedding extends to a map $A_n \xrightarrow{f} M$, and so defines a map of complexes $A \to D^{n+1}M$, which is f in degree n. This map obviously cannot extend to a map $B \to D^{n+1}M$ unless C_n is 0. Since the map $D^{n+1}M \to 0$ is in K, this shows that every map in K-proj is an injection.

Conversely, suppose we have a commutative diagram in $\mathrm{Ch}(\mathcal{A})$ as follows,

$$
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\
B & \xrightarrow{g} & Y
\end{array}
$$

where i is an injection and p is a surjection with injective kernel W. Since W is injective in $\mathrm{Ch}(\mathcal{A})$, there is a splitting $q\colon Y \to X$ of p. We have $p(qgi - f) = 0$, so, since W is injective, there is an extension $h\colon B \to W$ such that $hi = qgi - f$. Then $qg - h\colon B \to X$ is the desired lift. Hence i is in K-proj.

Now, we always have $(K$-proj$)$-inj $\supseteq K$. Conversely, suppose $p\colon X \to Y$ has the right lifting property with respect to all injections. Consider the map $D^n(Y_n) \to Y$ that is the identity in degree n. Since p has the right lifting property with respect to all injections, there is a lift $D^n Y \to X$ of this map. This shows that p is a split surjection in each dimension. Since the map $\ker p \to 0$ is a pullback of p, it too will have the right lifting property with respect to all injections, and so $\ker p$ is injective as an object of $\mathrm{Ch}(\mathcal{A})$. $\qquad\square$

We can now prove that I generates all injections.

Proposition 2.8. *Suppose $\mathcal{A}$ is a Grothendieck category. The class I-cof is the class of injections, and the class I-inj is the class of surjections whose kernel is an injective object of $\mathrm{Ch}(\mathcal{A})$.*

Proof. Let K denote the class of surjections whose kernel is injective. Applying Lemma 2.7, we see that $I \subseteq K$-proj, so I-cof $\subseteq (K$-proj$)$-cof $= K$-proj. Thus I-cof consists of injections. Furthermore, if we can show I-cof consists of all injections, Lemma 2.7 will show that I-inj is K, as required.

So suppose $i\colon A \to B$ is an injection. To show that $i \in I$-cof, we will show that i has the left lifting property with respect to I-inj. So suppose $p\colon X \to Y$ is in

I-inj, and we have a commutative diagram as follows.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\
B & \xrightarrow[\ g\ ]{} & Y
\end{array}
$$

Let T be the set of partial lifts of this diagram, so T is the set of all pairs (C, h), where C is a subcomplex of B containing $i(A)$ and $h\colon C \to X$ is a chain map such that $ph = g|_C$ and $hi = f$. Then T is a partially ordered set, where $(C, h) \le (C', h')$ if C' contains C and h' extends h. The set T is nonempty and we claim that every chain in T has an upper bound. Indeed, given a chain (C_i, h_i) in T, the colimit C of the C_i is still a subcomplex of B, by the AB5 condition, and the union of the h_i defines a lift on C. Thus there is a maximal element (M, h) of T. Suppose that M is not all of B, and choose a homogeneous element $x \in B$ that is not in M. Let Z be the smallest subcomplex (in $\mathrm{Ch}(\mathcal{A})$) of B containing x, so that $|Z| \le \gamma$ by Corollary 2.6. Let M' denote the subcomplex of B generated by M and x, so that we have the pushout diagram below.

$$
\begin{array}{ccc}
M \cap Z & \longrightarrow & Z \\
\downarrow & & \downarrow \\
M & \longrightarrow & M'
\end{array}
$$

Since the top horizontal map is in I, the bottom horizontal map is in I-cof. Hence there is a lift h' in the following diagram.

$$
\begin{array}{ccc}
M & \xrightarrow{\ h\ } & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
M' & \xrightarrow[\ g\ ]{} & Y
\end{array}
$$

This lift violates the maximality of (M, h), so we must have $M = B$. Hence $i \in I$-cof, as required. $\square$

Corollary 2.9. *Suppose $\mathcal{A}$ is a Grothendieck category. Then every injective object of $\mathrm{Ch}(\mathcal{A})$ is injectively fibrant and has no homology. Every map in I-inj is an injective fibration and a quasi-isomorphism.*

Proof. The second statement follows from the first. Indeed, a map in I-inj has the right lifting property with respect to all injections, so in particular is an injective fibration. If $p \in I$-inj, then $\ker p$ is an injective object of $\mathrm{Ch}(\mathcal{A})$, so has no homology by the first statement. Thus p is a homology isomorphism, by the long exact sequence.

Now suppose X is an injective object of $\mathrm{Ch}(\mathcal{A})$. Certainly X is injectively fibrant. To see that X has no homology, let Y denote the complex defined by $Y_n = X_n \oplus X_{n-1}$ with $d(x, y) = (dx + y, -dy)$. Then $X \to Y$ is an inclusion of complexes, so since X is injective, has a retraction $Y \to X$. This retraction is equivalent to a contracting homotopy of X, so in particular X has no homology. $\square$

To complete the proof of Theorem 2.2, we must show that J-cof is the class of injective quasi-isomorphisms, from which it will follow that J-inj is the class of injective fibrations. We begin with the following crucial, but technical, lemma.

Lemma 2.10. *Suppose $\mathcal{A}$ is a Grothendieck category. Suppose $i\colon A \to B$ is an injective quasi-isomorphism in $\mathrm{Ch}(\mathcal{A})$. For every subcomplex C of B in $\mathrm{Ch}(\mathcal{A})$ with $|C| \leq \gamma$, there is a subcomplex D of B in $\mathrm{Ch}(\mathcal{A})$ containing C such that $|D| \leq \gamma$ and $i\colon D \cap A \to D$ is a weak equivalence.*

Proof. The failure of $C \cap A \to C$ to be a quasi-isomorphism is measured by $H_*(C/C \cap A)$. Suppose for the moment that for every homogeneous element x of $H_*(C/C \cap A)$, we can find a subcomplex $C(x)$ containing C such that $|C(x)| \leq \gamma$ and the map $H_*(C/C \cap A) \to H_*(C(x)/C(x) \cap A)$ sends x to 0. Since $|C| \leq \gamma$, the R-module homology of $C/C \cap A$ has size $\leq \gamma$. But then Lemma 2.5 assures that $|H_*(C/C \cap A)| \leq \gamma$, so have $\leq \gamma$ choices for x. We can therefore take the union of all the $C(x)$ to form a new subcomplex FC with $|FC| \leq \gamma$ (using Lemma 2.5 again), such that the induced map $H_*(C/C \cap A) \to H_*(FC/FC \cap A)$ is the zero map.

Now iterate this construction to form a sequence $F^n C$, and let D be the colimit of all the $F^n C$. Then $|D| \leq \gamma$, by Lemma 2.5. Note that $D/D \cap A$ is the colimit of the $F^n C/F^n C \cap A$, by commuting colimits. Lemma 1.5 then shows that $H_*(D/D \cap A) = 0$, as required.

To complete the proof, we must construct the complex $C(x)$. The construction we give is fairly complicated; we do not know if there is a simpler one. Let us denote the R-module homology of a complex X by $\widetilde{H}(X)$ and let us denote the torsion submodule of an R-module M by tM. Then the class x is represented by a homomorphism $f\colon \mathfrak{a} \to \widetilde{H}_n(C/C \cap A)/t\widetilde{H}_n(C/C \cap A)$, for some left ideal $\mathfrak{a}$ of R such that $R/\mathfrak{a}$ is in the torsion theory. The class x must map to 0 in $H_n(B/A)$, since $A \to B$ is a quasi-isomorphism. This means that there is a subideal $\mathfrak{b}$ of $\mathfrak{a}$ with $R/\mathfrak{b}$ also in the torsion theory, such that the composite

$$\mathfrak{b} \xrightarrow{f} \widetilde{H}_n(C/C \cap A)/t\widetilde{H}_n(C/C \cap A) \to \widetilde{H}_n(B/A)/t\widetilde{H}_n(B/A)$$

is the zero map. We need to construct $C(x)$ so that this map is already the zero map when $C(x)$ replaces B.

For each $y \in \mathfrak{b}$, choose an element z_y in C_n with $dz_y \in A_n$ whose homology class $[z_y]$ is a representative for $f(y)$. Since the above composite is 0, there is an ideal $\mathfrak{c}_y$ such that $R/\mathfrak{c}_y$ is in the torsion theory, and $\mathfrak{c}_y[z_y] = 0$ in $\widetilde{H}_n(B/A)$. This means that, for every $w \in \mathfrak{c}_y$, there is an element $v_{w,y} \in B_{n+1}$ such that $wz_y - dv_{w,y} \in A_n$. We define $C(x)$ to be the smallest subcomplex of B containing C and all the $v_{w,y}$. It is clear from the construction that x goes to 0 in $H_n(C(x)/C(x) \cap A)$. Since there are $\leq |R| \leq \gamma$ choices for w and y, the smallest subcomplex of R-modules containing C and the $v_{w,y}$ has size $\leq \gamma$. Lemma 2.5 then shows $|C(x)| \leq \gamma$, as required. $\qquad\square$

With this lemma in hand, it is now not difficult to show that J-cof is the class of injective quasi-isomorphisms.

Proposition 2.11. *Suppose $\mathcal{A}$ is a Grothendieck category. Then the class J-cof consists of the injective quasi-isomorphisms, and the class J-inj consists of the injective fibrations.*

By the recognition theorem [Hov98, Theorem 2.1.19], this proposition completes the proof of Theorem 2.2.

Proof. The second statement is an immediate corollary of the first. By Corollary 1.7, the maps of J-cof are injective quasi-isomorphisms. Now suppose $i\colon A \to B$ is an injective quasi-isomorphism. To show that $i \in J$-cof, we show that i has the left lifting property with respect to J-inj. So suppose p is in J-inj, and we have a commutative diagram as follows.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\
B & \xrightarrow[\ g\ ]{} & Y
\end{array}
$$

Let T denote the set of partial lifts (C, h), where C is a subcomplex of B containing iA such that the map $i\colon A \to C$ is a quasi-isomorphism, and $h\colon C \to X$ is a partial lift in our diagram. Then T is obviously partially ordered and nonempty. Proposition 1.6 and the argument used in the proof of Proposition 2.8 imply that a chain in T has an upper bound. Zorn's lemma then gives us a maximal element (M, h) of T. Suppose M is not all of B, and choose an element x in B but not in M. Let C denote the subcomplex of B generated by x, so $|C| \le \gamma$ by Corollary 2.6. Since $M \to B$ is a quasi-isomorphism, Lemma 2.10 implies that there is a complex D containing C such that $|D| \le \gamma$ and the map $D \cap M \to D$ is a quasi-isomorphism. Let N denote the subcomplex of B generated by M and D. Then the map $M \to N$ is in J-cof, since it is a pushout of $D \cap M \to D$. Since $p \in J$-inj, there is an extension of h to N, contradicting the maximality of (M, h). Therefore we must have had $M = B$, and so $i \in J$-cof, as required. $\square$

To complete the description of the injective model structure, we would like to characterize the injective fibrations. This characterization is precisely the same as the corresponding characterization in the category of chain complexes of modules, found in [Hov98, 2.3.16–20], with the same proofs.

Proposition 2.12. *Suppose $\mathcal{A}$ is a Grothendieck category. Then a map $p \in \mathrm{Ch}(\mathcal{A})$ is an injective fibration if and only if it is a split surjection in each degree with injectively fibrant kernel. Any injectively fibrant complex is a complex of injective objects, and any bounded above complex of injective objects is injectively fibrant.*

We now discuss the functoriality of the injective model structure.

Proposition 2.13. *Suppose $F\colon \mathcal{A} \to \mathcal{B}$ is a functor between Grothendieck categories, with right adjoint U. Then F induces a Quillen adjunction $F\colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$ between the injective model structures if and only if F is exact.*

Proof. If F is exact, then clearly F preserves injections and all quasi-isomorphisms, so preserves cofibrations and trivial cofibrations. Conversely, suppose F preserves cofibrations and trivial cofibrations. We can think of an exact sequence as a complex X with no homology, so $0 \to X$ is a trivial cofibration. Then $0 \to FX$ must also be a trivial cofibration, so F must be exact. $\square$

Note that F being exact is equivalent to U being additive and preserving injectives. This proposition is expected, but not very satisfying. It means we cannot use the injective model structure to form any interesting total left derived functors,

since such a total left derived functor would be defined by first replacing an object by a cofibrant object weakly equivalent to it, and every object is already cofibrant. We can use the injective model structure to form some right derived functors.

In fact, we can use it to form more right derived functors than one might first expect. The construction of the total right derived functor of U [Hov98, Definition 1.3.6] only requires that U preserve weak equivalences between injectively fibrant objects. But a quasi-isomorphism between injectively fibrant chain complexes is in fact a chain homotopy equivalence (since every object is cofibrant). So in order to construct the right derived functor of U, we only need to insure that U preserve chain homotopy. For this, all we require is that U be additive.

We have then proved the following proposition.

Proposition 2.14. *Suppose $U \colon \mathcal{A} \to \mathcal{B}$ is an additive functor between Grothendieck categories. Then the total right derived functor $RU \colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ of U exists.*

This recovers the usual right derived functors of U: if $X \in \mathcal{A}$, we have $(R^i U)(X) = H_i((RU)X)$. The functor RU is of course calculated by replacing X by an injective resolution (or an injectively fibrant approximation if X is a complex), then applying U. In particular, if f is a map of ringed spaces, we recover the total right derived functor Rf_* between complexes of sheaves in this way.

3. AN ALTERNATIVE APPROACH

We have already discussed the drawbacks of the injective model structure on a Grothendieck category $\mathcal{A}$. In this section, we offer another approach; we will apply it to the category of sheaves on a ringed space satisfying a hypthesis related to finite global dimension in the next section. Though this is our only application of this approach, we present the method in a general fashion in the hope that it may find other applications. This approach is based on the standard projective model structure when $\mathcal{A} = R$-Mod for some ring R, and generalizations of it considered by Christensen in [Chr98]. Recall from [Hov98, Section 2.3] that the projective model structure on $\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A} = R$-Mod for some ring R, is a cofibrantly generated model structure, with generating cofibrations $I = \{S^{n-1}R \to D^n R\}$ and generating trivial cofibrations $J = \{0 \to D^n R\}$. Here n runs through all integers, $S^{n-1}M$ is the complex whose only nonzero object is M in dimension $n-1$, and $D^n M$ is the complex whose only nonzero objects are M in dimensions n and $n-1$. Our plan is to replace the map $0 \to R$ by a set of monomorphisms $\mathcal{M}$.

Definition 3.1. Suppose $\mathcal{M}$ is a set of monomorphisms in a Grothendieck category $\mathcal{A}$. Let $\mathcal{F}$ denote the set of codomains of the maps of $\mathcal{M}$. We will say that $\mathcal{M}$ is *pointed* if $0 \in \mathcal{F}$ and, if $F \in \mathcal{F}$, then $0 \to F$ is in $\mathcal{M}$. Define J to be the set of all $D^n f$, where n is an integer and $f \in \mathcal{M}$. Define I to be the union of J and the maps $S^{n-1}F \to D^n F$ for $F \in \mathcal{F}$ and n an integer. Then define a map p to be a $\mathcal{M}$-*fibration* if p is in J-inj, define p to be a $\mathcal{M}$-*cofibration* if p is in I-cof.

If $\mathcal{M}$ consists only of the maps $0 \to F$ for $F \in \mathcal{F}$, then we recover the definitions of [Chr98].

Our goal is to determine conditions on $\mathcal{M}$ under which the quasi-isomorphisms, the $\mathcal{M}$-cofibrations, and the $\mathcal{M}$-fibrations determine a model structure on $\mathrm{Ch}(\mathcal{A})$. We use the recognition theorem [Hov98, Theorem 2.1.19]. Since the maps of J

are injective quasi-isomorphisms in I-cof, the maps of J-cof will also be, by Corollary 1.7. Hence we need to show that the maps of I-inj coincide with the maps that are both $\mathcal{M}$-fibrations and quasi-isomorphisms.

We begin by characterizing the $\mathcal{M}$-fibrations.

Definition 3.2. Suppose $\mathcal{M}$ is a pointed set of monomorphisms in a Grothendieck category $\mathcal{A}$. Define an object X of $\mathcal{A}$ to be $\mathcal{M}$-*flasque* if $\mathcal{A}(f, X)$ is surjective for all $f \in \mathcal{M}$.

This definition is a generalization of the usual notion of flasque, or flabby, sheaves. We will discuss this in detail in the next section.

Let us denote the category of chain complexes of abelian groups by $\mathrm{Ch}(\mathbb{Z})$.

Proposition 3.3. *Suppose $\mathcal{M}$ is a pointed set of monomorphisms in a Grothendieck category $\mathcal{A}$. Then a map $p\colon X \to Y$ in $\mathrm{Ch}(\mathcal{A})$ is a $\mathcal{M}$-fibration if and only if $\mathcal{A}(F, p)$ is a surjection in $\mathrm{Ch}(\mathbb{Z})$ for all F in $\mathcal{F}$ and $\ker p$ is dimensionwise $\mathcal{M}$-flasque.*

In particular, if $\mathcal{F}$ is a set of generators for $\mathcal{A}$, then $\mathcal{M}$-fibrations are surjective. To see this, consider the map from Y_n into the cokernel of p_n.

Proof. Adjointness implies that p has the right lifting property with respect to $D^n B \xrightarrow{D^n f} D^n C$ if and only if the map

$$\mathcal{A}(C, X_n) \to \mathcal{A}(C, Y_n) \times_{\mathcal{A}(B, Y_n)} \mathcal{A}(B, X_n)$$

is surjective. Applying this when f is the map $0 \to F$ for $F \in \mathcal{F}$, we find that, if p is a $\mathcal{M}$-fibration, then $\mathcal{A}(F, p)$ is surjective. Furthermore, if p is a $\mathcal{M}$-fibration, then $\ker p \to 0$ is in J-inj. Applying the above criterion, we find that $\ker p$ is dimensionwise $\mathcal{M}$-flasque.

Conversely, suppose $\mathcal{A}(F, p)$ is a surjection for all $F \in \mathcal{F}$ and $K = \ker p$ is dimensionwise $\mathcal{M}$-flasque. Suppose $f\colon B \to C$ is in $\mathcal{M}$. We have an exact sequence

$$0 \to \mathcal{A}(B, K_n) \to \mathcal{A}(B, X_n) \to \mathcal{A}(B, Y_n)$$

and a similar exact sequence that is in fact short exact when B is replaced by C. By pulling back the exact sequence for B through the map $\mathcal{A}(f, Y_n)$, we obtain the following commutative diagram whose top row is short exact and whose bottom row only misses being short exact because the right map is not necessarily surjective.

$$\begin{array}{ccccc}
\mathcal{A}(C, K_n) & \longrightarrow & \mathcal{A}(C, X_n) & \longrightarrow & \mathcal{A}(C, Y_n) \\
{\scriptstyle \mathcal{A}(f, K_n)}\downarrow & & \downarrow & & \| \\
\mathcal{A}(B, K_n) & \longrightarrow & \mathcal{A}(C, Y_n) \times_{\mathcal{A}(B, Y_n)} \mathcal{A}(B, X_n) & \longrightarrow & \mathcal{A}(C, Y_n)
\end{array}$$

Since K is dimensionwise $\mathcal{M}$-flasque, the left-hand vertical map is surjective. A standard diagram chase, as in the snake lemma, then show that the middle vertical map is surjective, so p is a $\mathcal{M}$-fibration. $\square$

Proposition 3.4. *Suppose $\mathcal{M}$ is a pointed set of monomorphisms in a Grothendieck category $\mathcal{A}$. Suppose in addition that the set $\mathcal{F}$ of codomains of $\mathcal{M}$ generates $\mathcal{A}$. Then every map of complexes $p\colon X \to Y$ in I-inj is both a $\mathcal{M}$-fibration and a quasi-isomorphism.*

Proof. Recall that the functor $S^{n-1} \colon \mathcal{A} \to \mathrm{Ch}(\mathcal{A})$ is left adjoint to the functor that takes X to $Z_{n-1}X$, the cycles in X_{n-1}. This implies that p is in I-inj if and only if it is a $\mathcal{M}$-fibration and the map

$$\mathcal{A}(F, X_n) \to \mathcal{A}(F, Y_n) \times_{\mathcal{A}(F, Z_{n-1}Y)} \mathcal{A}(F, Z_{n-1}X)$$

is surjective for all n and $F \in \mathcal{F}$. Let $K = \ker p$. If $p \in I$-inj, then the map $K \to 0$ is as well. Hence the map $\mathcal{A}(F, K_n) \to \mathcal{A}(F, Z_{n-1}K)$ is surjective for all n and all $F \in \mathcal{F}$. Since $\mathcal{F}$ is a set of generators for $\mathcal{A}$, this implies that the map $K_n \to Z_{n-1}K$ is surjective, and hence that K has no homology. A similar argument shows that p is surjective, and so the long exact sequence implies that p is a quasi-isomorphism. $\square$

If $\mathcal{F}$ is not a generating set for $\mathcal{A}$, we can still say that, if $p \in I$-inj, then $\mathcal{A}(F, p)$ is a surjective quasi-isomorphism for all $F \in \mathcal{F}$.

To complete the construction of our model structure, we need to know that every map that is both a $\mathcal{M}$-fibration and a quasi-isomorphism is in I-inj. We begin with a lemma.

Lemma 3.5. *Suppose $\mathcal{M}$ is a pointed set of monomorphisms in a Grothendieck category $\mathcal{A}$, and let $\mathcal{F}$ be the set of codomains of $\mathcal{M}$. Suppose $p \colon X \to Y$ is a map in $\mathrm{Ch}(\mathcal{A})$ such that $\mathcal{A}(F, p)$ is surjective for all $F \in \mathcal{F}$. Then p is in I-inj if and only if $\ker p \to 0$ is in I-inj.*

Proof. The only if implication is clear. Suppose $\mathcal{A}(F, p)$ is surjective for all $F \in \mathcal{F}$, and let $K = \ker p$. Suppose $K \to 0$ is in I-inj. In particular, this means that K is dimensionwise $\mathcal{M}$-flasque, so $p \in J$-inj. In order to show that p is in I-inj, we must show that, given $F \in \mathcal{F}$, a map $x \colon F \to Z_{n-1}X$, and a map $y \colon F \to Y_n$ such that $d \circ y = p \circ x$, there is a map $x' \colon F \to X_n$ such that $p \circ x' = y$ and $d \circ x' = x$. First choose $z \colon F \to X_n$ such that $p \circ z = y$, using the fact that $\mathcal{A}(F, p)$ is surjective. Then $p \circ (d \circ z - x) = 0$, so $dz - x \colon F \to Z_{n-1}K$. Since $K \to 0$ is in I-inj, there is a map $w \colon F \to K_n$ such that $d \circ w = d \circ z - x$. Now let $x' = z - w$. $\square$

Proposition 3.6. *Suppose $\mathcal{M}$ is a pointed set of monomorphisms in a Grothendieck category $\mathcal{A}$. Suppose that the set $\mathcal{F}$ of codomains of $\mathcal{M}$ generates $\mathcal{A}$, and, furthermore, suppose that if K is an acyclic, dimensionwise $\mathcal{M}$-flasque, complex and $F \in \mathcal{F}$, then $\mathcal{A}(F, K)$ is an acyclic complex of abelian groups. Then, if $p \colon X \to Y$ is a $\mathcal{M}$-fibration and quasi-isomorphism in $\mathrm{Ch}(\mathcal{A})$, then p is in I-inj.*

Proof. By Lemma 3.5, it suffices to show that $K = \ker p \to 0$ is in I-inj. But K is an acyclic dimensionwise flasque complex, and so $\mathcal{A}(F, K)$ is also acyclic. Hence the map $\mathcal{A}(F, K_n) \to \mathcal{A}(F, Z_{n-1}K)$ is surjective, and so $K \to 0$ is in I-inj. $\square$

We have proved the following theorem.

Theorem 3.7. *Suppose $\mathcal{M}$ is a pointed set of monomorphisms in a Grothendieck category $\mathcal{A}$ such that the set of codomains $\mathcal{F}$ of $\mathcal{M}$ forms a generating set of $\mathcal{A}$ and, for all acyclic, dimensionwise $\mathcal{M}$-flasque, complexes X and for all $F \in \mathcal{F}$, the complex $\mathcal{A}(F, X)$ is acyclic. Then $\mathrm{Ch}(\mathcal{A})$ is a proper cofibrantly generated model category, where the weak equivalences are the quasi-isomorphisms, the fibrations are the $\mathcal{M}$-fibrations, and the cofibrations are the $\mathcal{M}$-cofibrations.*

One interesting feature of the hypotheses of this theorem is that, if they are true for a given set of monomorphisms $\mathcal{M}$ with codomains $\mathcal{F}$, then they remain true if we expand $\mathcal{M}$ by adding any set of monomorphisms whose codomains are all in $\mathcal{F}$. So in fact we get many different model structures with the same weak equivalences, all relying on more or less stringent definitions of "flasque".

One might hope that we would still get a model structure on $\mathrm{Ch}(\mathcal{A})$ if we drop all hypotheses about the set of monomorphisms $\mathcal{M}$. The weak equivalences would have to change, probably to maps f such that $\mathcal{A}(F, f)$ is a quasi-isomorphism for all $F \in \mathcal{F}$. With this definition, an appropriately modified version of Proposition 3.6 does hold. However, we do not know if the maps of J-cof are weak equivalences with this definition.

4. Generators of finite projective dimension

In this section, we apply the method of the previous section to construct a new model structure on $\mathrm{Ch}(\mathcal{A})$, when $\mathcal{A}$ is a Grothendieck category with generators of finite projective dimension. Recall that an object B is said to have *finite projective dimension* if there is an integer n_0 such that $\mathrm{Ext}_{\mathcal{A}}^n(B, C) = 0$ for all $n \geq n_0$ and all object C of $\mathcal{A}$. One normally thinks of an object of finite projective dimension as being the 0th homology group of a finite complex of projectives, but this will not be true unless there are enough projectives in the category. In the categories we are interested in, this is almost never true.

Nevertheless, objects of finite projective dimension are useful in constructing a model structure because of the following lemma.

Lemma 4.1. *Suppose $\mathcal{A}$ is a Grothendieck category, $F \in \mathcal{A}$ has finite projective dimension, and $X \in \mathrm{Ch}(\mathcal{A})$ is an acyclic complex such that $\mathrm{Ext}_{\mathcal{A}}^i(F, X_n) = 0$ for all $i > 0$ and all n. Then $\mathcal{A}(F, X)$ is still acyclic.*

Proof. Since X is acyclic, we have a short exact sequence

$$Z_n X \to X_n \to Z_{n-1} X.$$

Since $\mathrm{Ext}_{\mathcal{A}}^i(F, X_n) = 0$ for all $i > 0$, this gives us an exact sequence

$$0 \to \mathcal{A}(F, Z_n X) \to \mathcal{A}(F, X_n) \to \mathcal{A}(F, Z_{n-1} X) \to \mathrm{Ext}_{\mathcal{A}}^1(F, Z_n X) \to 0$$

and isomorphisms $\mathrm{Ext}_{\mathcal{A}}^i(F, Z_{n-1} X) \cong \mathrm{Ext}_{\mathcal{A}}^{i+1}(F, Z_n X)$ for $i > 0$. Thus

$$\mathrm{Ext}_{\mathcal{A}}^1(F, Z_n X) \cong \mathrm{Ext}_{\mathcal{A}}^{m+1}(F, Z_{m+n} X)$$

for all $m \geq 0$. Since F has finite projective dimension, this implies $\mathrm{Ext}_{\mathcal{A}}^1(F, Z_n X) = 0$ for all n. It follows that $\mathcal{A}(F, X)$ is acyclic. $\square$

Theorem 4.2. *Suppose $\mathcal{A}$ is a Grothendieck category with a set of generators $\mathcal{F}$, each element of which has finite projective dimension. Let $\mathcal{M}$ denote the set of inclusions $A \to F$ of subobjects of objects $F \in \mathcal{F}$. Then there is a proper cofibrantly generated model structure on $\mathrm{Ch}(\mathcal{A})$, where the weak equivalences are the quasi-isomorphisms, the fibrations are the dimensionwise split surjections with dimensionwise injective kernel, and the cofibrations are the $\mathcal{M}$-fibrations.*

Proof. Note first that the $\mathcal{M}$-flasque objects of $\mathcal{A}$ coincide with the injective objects, by [Ste75, Prop. V.2.9]. Lemma 4.1 implies that if X is an acyclic, dimensionwise injective, complex, then $\mathcal{A}(F, X)$ is acyclic for all $F \in \mathcal{F}$. Hence Theorem 3.7 gives us a model structure. Any $\mathcal{M}$-fibration is a surjection with dimensionwise injective

kernel, by Proposition 3.3, and therefore must be a dimensionwise split surjection. Conversely, a dimensionwise split surjection with dimensionwise injective kernel certainly satisfies the conditions of Proposition 3.3, so is a $\mathcal{M}$-fibration. □

This model structure is related to the injective model structure; the identity functor from this model structure to the injective model structure is a Quillen equivalence. It appears to be new even when $\mathcal{A}$ is the category of modules over a ring R. The generating cofibrations and trivial cofibrations in this model structure are explicit, and the fibrations are easier to understand than the injective fibrations. On the other hand, we know nothing about the cofibrations in this model structure.

In general, this model structure is poorly behaved with respect to functors of abelian categories. If F is an additive functor with right adjoint U, then U will preserve fibrations in this model structure if an only if U preserves injectives, which is equivalent to F being exact. But this is not enough to conclude that F induces a Quillen functor; we must also know that U preserves acyclic complexes of injectives. This will happen if U is exact, but may happen in some other cases as well.

We now consider an interesting example of this model structure. Suppose S is a noetherian scheme. We say that S *has enough locally frees* if every coherent sheaf on S is a quotient of a locally free sheaf of finite rank. For example, a noetherian, integral, separated, locally factorial scheme has enough locally frees by a result of Kleiman [Har77, Ex. III.6.8].

Proposition 4.3. *Suppose S is a noetherian scheme with enough locally frees. In addition, suppose that either S is finite-dimensional or is separated. Then the locally free sheaves of finite rank are generators of finite projective dimension for the category $\mathrm{QCo}(S)$ of quasi-coherent sheaves on X.*

Proof. We first show that the locally frees generate $\mathrm{QCo}(S)$. Deligne [Har66, Appendix, Prop. 2] shows that every quasi-coherent sheaf is a colimit of finitely presented sheaves. On a noetherian scheme, finitely presented sheaves are coherent, and thus, since S has enough locally frees, are quotients of locally free sheaves of finite rank.

Now let F be a locally free sheaf of finite rank, and C a quasi-coherent sheaf of $\mathcal{O}$-modules on S. By the corollary to [Gro57, Prop. 4.2.3], we have

$$\mathrm{Ext}^i_{\mathcal{O}\text{-Mod}}(F, C) \cong H^i(S; \mathrm{Hom}(F, C))$$

where Hom denotes sheaf Hom and the cohomology groups are sheaf cohomology. If S is finite-dimensional, we can apply Grothendieck's vanishing theorem [Har77, Theorem III.2.7] to conclude that these cohomology groups are 0 for large enough i. If S is separated, then we can apply [Har77, Ex. III.4.8] to reach the same conclusion, using the fact that $\mathrm{Hom}(F, C)$ is quasi-coherent.

This does not complete the proof, because these are Ext groups in $\mathcal{O}$-Mod rather than in $\mathrm{QCo}(S)$. However, these two possibly different Ext groups in fact coincide, because the exact inclusion functor $\mathrm{QCo}(S) \to \mathcal{O}$-Mod has a right adjoint and left inverse Q [SGA6, p. 187] whenever S is quasi-compact and quasi-separated, as any noetherian scheme is. In detail, given a quasi-coherent sheaf C, we can first take an injective resolution I_* of C in $\mathcal{O}$-Mod and apply Q to get a complex of injectives QI_* in $\mathrm{QCo}(S)$. We claim that QI_* is still exact. To see this, consider the short exact sequence

$$0 \to C \to I_0 \to ZI_1 \to 0.$$

After we apply Q, we get a long exact sequence involving the derived functors R^iQ of Q. However, $R^iQC = 0$ for $i > 0$, by the last paragraph of [SGA6, p. 189]. Furthermore, $R^iQI_0 = 0$ for $i > 0$ because I_0 is injective. It follows that $(R^iQ)ZI_1 = 0$ for $i > 0$ as well. Repeating this argument on the short exact sequence

$$0 \to ZI_1 \to I_1 \to ZI_2 \to 0,$$

we find that $(R^iQ)ZI_2 = 0$ for $i > 0$, and, by induction, that $(R^iQ)ZI_m = 0$ for all m and $i > 0$. Hence QI_* is still exact, and so is an injective resolution of C in QCo(S).

Applying QCo$(S)(B, -)$ to QI_* and using adjointness, we find that, if B and C are both quasi-coherent, then $\mathrm{Ext}^i_{\mathrm{QCo}(S)}(B, C) = \mathrm{Ext}^i_{\mathcal{O}\text{-Mod}}(B, C)$, completing the proof.   □

Hence, as a corollary to Proposition 4.3 and Theorem 4.2, we get the following theorem.

Theorem 4.4. *Suppose S is a noetherian scheme with enough locally frees, and suppose that S is either finite-dimensional or separated. Then there is a proper, cofibrantly generated, model structure on the category $\mathrm{Ch}_{\mathrm{QCo}}(S)$ of unbounded complexes of quasi-coherent sheaves, where the weak equivalences are the quasi-isomorphisms and the fibrations are the dimensionwise split surjections with dimensionwise injective kernel.*

Let us call this model strucure the *locally free model structure*. We do not understand the cofibrations in the locally free model structure, though we point out that S^nF is cofibrant for any locally free F, and D^nA is cofibrant for any coherent sheaf A. If $f \to S \to T$ is a map between schemes satisfying the hypotheses of Theorem 4.4, then the functor $f^*\colon \mathrm{QCo}(T) \to \mathrm{QCo}(S)$ will induce a Quillen functor between the locally free model structures if and only if f^* is exact; we have already seen that this is necessary, and it is sufficient since f^* preserves locally free sheaves of finite rank.

Despite these drawbacks, the locally free model structure does gives some information about the derived category $D(\mathrm{QCo}(S))$.

Corollary 4.5. *Suppose S is a noetherian scheme with enough locally frees, and either S is finite-dimensional or separated. Then the locally free sheaves of finite rank form a set of small weak generators for the derived category $D(\mathrm{QCo}(S))$.*

Proof. The fact that the locally free sheaves form a set of weak generators follows from [Hov98, Section 7.3]. To see that they are small, in the triangulated sense, we use the result of [Hov98, Section 7.4]. We must then show that, if F is a locally free sheaf of finite rank, the functor $\mathrm{QCo}(S)(F, -)$ preserves all transfinite compositions. Since we are on a noetherian scheme, we can take the transfinite composition in the category of presheaves [Har77, Ex. II.1.11]. It is then easy to check the desired result.   □

In case S is a quasi-compact, quasi-separated scheme, we can use the right adjoint Q to the inclusion $\mathrm{QCo}(S) \to \mathcal{O}\text{-Mod}$ to show that $\mathrm{QCo}(S)$ is a closed symmetric monoidal category under the tensor product. Thus $\mathrm{Ch}_{\mathrm{QCo}}(S)$ is also a closed symmetric monoidal category. It would be preferable, then, to have a model structure on $\mathrm{Ch}_{\mathrm{QCo}}(S)$ that is compatible with the closed symmetric monoidal structure, in the sense of [Hov98, Chapter 4]. This compatiblity condition is discussed before

Theorem 5.6 below. Unfortunately, the locally free model structure is not compatible with the tensor product.

Despite this, it is known that $D(\mathrm{QCo}(S))$ is a symmetric monoidal triangulated category, at least when S is a finite-dimensional noetherian scheme. Indeed, Lipman [Lip98, Section 2.5]shows that $D(\mathcal{O}\text{-Mod})$ is a symmetric monoidal triangulated category. But $D(\mathrm{QCo}(S))$ is equivalent to the full subcategory of $D(\mathcal{O}\text{-Mod})$ consisting of complexes with quasi-coherent cohomology, when S is a finite-dimensional noetherian scheme, by [SGA6, p. 191], and the inclusion $D(\mathrm{QCo}(S)) \to D(\mathcal{O}\text{-Mod})$ has a right adjoint given by the right derived functor of Q. It follows from this that $D(\mathrm{QCo}(S))$ is a symmetric monoidal triangulated category.

Furthermore, locally free sheaves of finite rank F are strongly dualizable in $D(\mathrm{QCo}(S))$. Recall that this means that the natural map

$$\mathrm{Hom}(F, \mathcal{O}) \otimes X \to \mathrm{Hom}(F, X)$$

is an isomorphism, where of course both the Hom and the tensor have to be interpreted in $D(\mathrm{QCo}(S))$, so are really derived versions. This follows from the corresponding fact in $\mathcal{O}\text{-Mod}$ itself, and the fact that locally free sheaves are flat.

In the language of [HPS97], then, we have proved the following corollary.

Corollary 4.6. *Suppose S is a finite-dimensional noetherian scheme with enough locally frees. Then the category $D(\mathrm{QCo}(S))$ is an unital algebraic stable homotopy category, where the generators are the locally free sheaves of finite rank.*

5. The flat model structure on sheaves

In this section, we apply the method of Theorem 3.7 to the category $\mathcal{O}\text{-Mod}$ of sheaves over a ringed space $(S, \mathcal{O})$. In this case, there is a standard set of generators; namely, the sheaves $\mathcal{O}_U$ for U an open set of S. Recall that $\mathcal{O}_U$ is the sheafification of the presheaf that assigns V to $\mathcal{O}(V)$ if $V \subseteq U$, and to 0 otherwise. The stalk of $\mathcal{O}_U$ at x is 0 if $x \notin U$, and is $\mathcal{O}_x$ if $x \in U$. We have $\mathcal{O}\text{-Mod}(\mathcal{O}_U, X) \cong X(U)$, which implies easily that the $\mathcal{O}_U$ form a generating set for $\mathcal{O}\text{-Mod}$.

Note that, if $V \subseteq U$, there is a natural monomorphism $\mathcal{O}_V \to \mathcal{O}_U$ corresponding to $1 \in \mathcal{O}_U(V)$. Thus, we take the set of monomorphisms $\mathcal{M}$ of the previous section to consist of these natural monomorphisms. One can then easily check that a sheaf X is $\mathcal{M}$-flasque if and only if the restriction maps $X(U) \to X(V)$ are surjective whenever $V \subseteq U$, corresponding to the usual notion of a flasque sheaf.

To apply Theorem 3.7 we need to know that, if X is an acyclic complex of flasque sheaves, then $\mathcal{O}\text{-Mod}(\mathcal{O}_U, X)$ is also acyclic; *i.e.*that X is acyclic as a complex of *presheaves*. Unfortunately, this need not always be true. Amnon Neeman has constructed a complex X of injective sheaves on infinite-dimensional real projective space whose sheaf cohomology is trivial, but whose presheaf cohomology is nontrivial. The example is a bit complicated, but is closely related to the example in [Hov98, Remark 2.3.18].

We therefore need a hypothesis on our ringed space to apply Theorem 3.7.

Definition 5.1. Define a ringed space $(S, \mathcal{O})$ to have finite global dimension if there is an integer $n > 0$ such that the sheaf cohomology $H^n(X) = 0$ for all $\mathcal{O}$-modules X. Define $(S, \mathcal{O})$ to have finite hereditary global dimension if every open ringed subspace $(U, \mathcal{O}|_U)$ has finite global dimension.

We then get the following theorem.

Theorem 5.2. *Suppose* $(S, \mathcal{O})$ *is a ringed space with finite hereditary global dimension. Then there is a cofibrantly generated proper model structure on* $\mathrm{Ch}(\mathcal{O}\text{-}Mod)$, *called the* flat model structure, *where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections with dimensionwise flasque kernel.*

Proof. We apply Theorem 3.7, taking the set $\mathcal{M}$ to be the canonical inclusions $\mathcal{O}_V \to \mathcal{O}_U$. We use Lemma 4.1. One can easily check that $\mathrm{Ext}^i_{\mathcal{O}}(\mathcal{O}_U, B) = H^i(U; B|_U)$; this is essentially the definition of sheaf cohomology. In particular, S has finite hereditary global dimension if and only if each $\mathcal{O}_U$ has finite projective dimension. Also, since the restriction of a flasque sheaf is still flasque and flasque sheaves have no cohomology, $\mathrm{Ext}^i(\mathcal{O}_U, X_n) = 0$ if $i > 0$ and X is a complex of flasque sheaves. So Lemma 4.1 applies, and Theorem 3.7 gives us the desired model structure.

The characterization of fibrations in Proposition 3.3 translates into surjections of *presheaves* with dimensionwise flasque kernel. However, sheaf surjections with flasque kernel are also presheaf surjections, so we get the claimed characterization of fibrations.   □

The author knows of two cases when ringed spaces are guaranteed to have finite hereditary global dimension.

Proposition 5.3. *Suppose* $(S, \mathcal{O})$ *is a ringed space.*
1. *If S is a finite-dimensional noetherian space, then $(S, \mathcal{O})$ has finite hereditary global dimension.*
2. *If S is a finite-dimensional locally compact topological manifold that is countable at infinity, in particular if S is a finite-dimensional compact manifold, then $(S, \mathcal{O})$ has finite hereditary global dimension.*

Proof. Part 1 follows from the vanishing theorem [Har77, Theorem III.2.7] of Grothendieck, since an open subspace of a finite-dimensional noetherian space is still a finite-dimensional noetherian space. Part 2 is an immediate consequence of [KS90, Proposition 3.2.2].   □

We now discuss the cofibrations in the flat model structure. Recall that the category $\mathcal{O}\text{-Mod}$ is a closed symmetric monoidal category. The monoidal structure is given by the tensor product $X \otimes_{\mathcal{O}} Y$, which we will always denote by $X \otimes Y$. This is defined by forming the obvious presheaf tensor product, and sheafifying. On each stalk, the tensor product is the ordinary tensor product of modules. In particular, a sheaf F is flat if and only if each stalk F_x is flat as a $\mathcal{O}_x$-module; hence, the sheaves $\mathcal{O}_U$ are flat. The closed structure is given by the sheaf Hom; $\mathrm{Hom}(X, Y)(U) = \mathcal{O}|_U\text{-Mod}(X|_U, Y|_U)$. These structures extend to complexes in the usual way, making $\mathrm{Ch}(\mathcal{O}\text{-Mod})$ into a closed symmetric monoidal category. This works for any symmetric monoidal additive category, as described in [HPS97, Section 9.2]

Definition 5.4. Suppose $\mathcal{A}$ is a symmetric monoidal abelian category. Define a complex $F \in \mathrm{Ch}(\mathcal{A})$ to be *DG-flat* if each F_n is flat, and, for any acyclic complex K, the complex $A \otimes K$ is also acyclic.

Proposition 5.5. *Suppose* $(S, \mathcal{O})$ *is a ringed space with finite hereditary global dimension. Then any cofibration in the flat model structure is a degreewise split monomorphism on each stalk, with DG-flat cokernel.*

We do not know if the converse to this proposition holds, nor even whether every DG-flat complex is cofibrant.

Proof. The maps of I are all degreewise split monomorphisms on each stalk. Every cofibration is a retract of a transfinite composition of pushouts of maps of I, by the small object argument [Hov98, Theorem 2.1.14]. Since retracts, transfinite compositions, and pushouts all commute with the operation of taking stalks and preserve split monomorphisms, every cofibration will be a degreewise split monomorphism on each stalk. The cokernel of a cofibration will of course be cofibrant, so to complete the proof it suffices to show that every cofibrant object is DG-flat.

Every cofibrant object A is a retract of the colimit of a transfinite sequence X_α, where each map $X_\alpha \to X_{\alpha+1}$ is a pushout of a map of I and $X_0 = 0$. Since colimits commute with tensor products and homology, it suffices to show that, if X_α is DG-flat, so is $X_{\alpha+1}$. On each stalk, the maps of I are degreewise split monomorphisms with degreewise flat cokernel, so the same will be true of $X_\alpha \to X_{\alpha+1}$. Thus, if X_α is a complex of flat sheaves, so is $X_{\alpha+1}$.

Now suppose K is an acyclic complex and f is a map of I. Again, since the maps of I are degreewise split monomorphisms on each stalk, the map $f \otimes K$ will still be injective. Thus the map $X_\alpha \otimes K \to X_{\alpha+1} \otimes K$ will be injective. It therefore suffices to show that $f \otimes K$ is a quasi-isomorphism, by Corollary 1.4. In case f is of the form $D^n \mathcal{O}_V \to D^n \mathcal{O}_U$, both the domain and codomain of f are contractible. The same will be true of $f \otimes K$, so $f \otimes K$ will be a quasi-isomorphism. In case f is of the form $S^{n-1} \mathcal{O}_U \to D^n \mathcal{O}_U$, the codomain of $f \otimes K$ is contractible, so it suffices to show that $S^{n-1} \mathcal{O}_U \otimes K$ is acyclic. But, since $\mathcal{O}_U$ is flat, we have $H_m(S^{n-1} \mathcal{O}_U \otimes K) = H_{m-n+1}(K) \otimes \mathcal{O}_U$, so we are done. $\square$

In particular, it follows that cofibrations are pure monomorphisms, in the sense that, if f is a cofibration and K is an arbitrary complex, then $f \otimes K$ is still a monomorphism.

We now show that the flat model structure is compatible with the tensor product on $\mathrm{Ch}(\mathcal{O}\text{-Mod})$. To do this, we need to recall the definition of this compatibility. If $f \colon A \to B$ and $g \colon C \to D$ are maps in a cocomplete closed symmetric monoidal category, we denote the induced map

$$(A \otimes D) \amalg_{A \otimes C} (B \otimes C) \to B \otimes D$$

by $f \square g$. In case $\mathcal{C}$ is also a model category, we say that $\mathcal{C}$ is a *symmetric monoidal model category* if, whenever f and g are cofibrations, so is $f \square g$, and furthermore, if one of f or g is a trivial cofibration, so is $f \square g$. This is the condition needed to ensure that the homotopy category $\mathrm{Ho}\,\mathcal{C}$ is again closed symmetric monoidal, as explained in [Hov98, Chapter 4]. (Actually one also needs a condition on the unit, but this condition is unnecessary when the unit is cofibrant, as it is in the flat model structure).

Theorem 5.6. *Suppose $(S, \mathcal{O})$ is a ringed space with finite hereditary global dimension, and f and g are maps in $\mathrm{Ch}(\mathcal{O}\text{-Mod})$.*

(a) *If f is a flat cofibration and g is a monomorphism, then $f \square g$ is a monomorphism.*

(b) *If f and g are flat cofibrations, then so is $f \square g$.*

(c) *If f is a flat cofibration and g is an injective quasi-isomorphism, then $f \square g$ is a quasi-isomorphism.*

(d) *If f is a flat trivial cofibration and g is a monomorphism, then $f \square g$ is a quasi-isomorphism.*

Proof. As explained in [Hov98, Chapter 4], since monomorphisms and injective quasi-isomorphisms are closed under retracts, transfinite compositions, and pushouts, it suffices to check this theorem when the flat cofibration is in fact a map of I, and the flat trivial cofibration is a map of J. We begin with parts (a) and (c). Suppose that $g \colon A \to B$ is a monomorphism, and suppose f is the map $D^n \mathcal{O}_V \to D^n \mathcal{O}_U$. Let P denote the domain of $f \square g$, and suppose $x \in S$. If $x \in V$, then the stalk of P_m at x is $(B_{m-n} \oplus B_{m-n+1})_x$; if $x \in U \setminus V$, then the stalk of P at x is $(A_{m-n} \oplus A_{m-n+1})_x$; and if x is not in U, then the stalk of P at x is 0. The stalk of the codomain of $f \square g$ at x is $(B_{m-n} \oplus B_{m-n+1})_x$ if x is in U, and 0 otherwise, and the map $f \square g$ does the obvious thing on the stalks. Hence $f \square g$ is a monomorphism. Furthermore, the domain and codomain of f are contractible, so the same will be true for $f \square g$. Thus $f \square g$ will be a quasi-isomorphism, completing the proof of part (c).

To complete the proof of part (a), we must show that $f \square g$ is a monomorphism, where now f is the map $S^{n-1} \mathcal{O}_U \to D^n \mathcal{O}_U$. In this case, the stalk of the domain P of $f \square g$ at a point x is 0 if $x \notin U$, and otherwise is $(A_{m-n} \oplus B_{m-n+1})_x$. The stalk of the codomain of $f \square g$ at x is 0 if $x \notin U$, and otherwise is $(B_{m-n} \oplus B_{m-n+1})_x$. The map $f \square g$ does the obvious thing, and so is a monomorphism.

For part (c), we can assume f is the map $S^{n-1} \mathcal{O}_U \to D^n \mathcal{O}_U$. Then the codomain of $f \square g$ is contractible, so it suffices to show that the domain P of $f \square g$ has no homology. Since $g \colon A \to B$ is an injective quasi-isomorphism, and $\mathcal{O}_U$ is flat, $g \otimes S^{n-1} \mathcal{O}_U$ is also an injective quasi-isomorphism. Hence its pushout $A \otimes D^n \mathcal{O}_U \to P$ is also an injective quasi-isomorphism. Since $D^n \mathcal{O}_U$ is contractible, it follows that P has no homology.

Finally, for part (d), we can assume that both f and g are maps of I. To calculate $f \square g$ in this case, use the easily checked (on stalks) fact that $\mathcal{O}_U \otimes \mathcal{O}_V \cong \mathcal{O}_{U \cap V}$. It follows that

$$S^m \mathcal{O}_U \otimes S^n \mathcal{O}_V \cong S^{m+n} \mathcal{O}_{U \cap V}$$

and

$$S^m \mathcal{O}_U \otimes D^n \mathcal{O}_V \cong D^{m+n} \mathcal{O}_{U \cap V}$$

and that $D^m \mathcal{O}_U \otimes D^n \mathcal{O}_V$ is an amalgamation of $D^{m+n-1} \mathcal{O}_{U \cap V}$ and $D^{m+n} \mathcal{O}_{U \cap V}$. With these identities in hand, the proof is a calculation we leave to the reader. $\quad\square$

Corollary 5.7. *Suppose $(S, \mathcal{O})$ is a ringed space with finite hereditary global dimension. Then the flat model structure makes $\mathrm{Ch}(\mathcal{O}\text{-Mod})$ into a symmetric monoidal model category. Furthermore, if A is cofibrant, then the functor $A \otimes -$ preserves quasi-isomorphisms. Therefore, to calculate the derived tensor product up to isomorphism, it suffices to replace* one *of the factors by a cofibrant complex quasi-isomorphic to it.*

Proof. The fact that $\mathrm{Ch}(\mathcal{O}\text{-Mod})$ is a symmetric monoidal model category is immediate from Theorem 5.6. Suppose A is cofibrant. Then $A \otimes -$ preserves trivial cofibrations in any symmetric monoidal model category. So, in order to see that $A \otimes -$ preserves quasi-isomorphisms, it suffices to show that, if p is a trivial fibration, then $A \otimes p$ is a quasi-isomorphism. Let K denote the kernel of p, so that

K is an acyclic complex. Since cofibrant objects are degreewise flat, $A \otimes p$ is still surjective, and its kernel is $A \otimes K$. Since cofibrant objects are DG-flat, $A \otimes K$ is still acyclic, so the long exact sequence completes the proof. In general, the total derived functor of the tensor product is defined by $X \otimes^L Y = QX \otimes QY$, where QX (resp. QY) is a functorial cofibrant replacement for X (resp. Y). But, since the map $QX \otimes QY \to QX \otimes Y$ is a quasi-isomorphism, $X \otimes^L Y$ is isomorphic in the derived category to $QX \otimes Y$. $\qquad\square$

Note that Theorem 5.6 actually says not only that the flat model structure is symmetric monoidal, but also that the injective model structure is a module over the flat model structure, in the sense of [Hov98, Chapter 4].

We can also use Theorem 5.6 to conclude that the derived category of $\mathcal{O}$-modules is almost a unital algebraic stable homotopy category [HPS97].

Corollary 5.8. *Suppose $(S, \mathcal{O})$ is a ringed space such that S is a finite-dimensional noetherian space. Then the derived category of $\mathcal{O}$-modules is a symmetric monoidal triangulated category and $\{\mathcal{O}_U\}$ is a set of small weak generators.*

Proof. It is well-known that the derived category of any abelian category is triangulated, but this also follows, in a stronger sense of the word triangulated, from the results of [Hov98, Chapter 7]. We have already seen that the flat model caetgory is a symmetric monoidal model category, so the derived category is also closed symmetric monoidal in a way that is compatible with the triangulation (see [Hov98, Chapter 6], with one technical point dealt with by [Hov98, Corollary 5.6.10]). Since the flat model structure is cofibrantly generated, the cofibers of the generating cofibrations form a set of weak generators [Hov98, Section 7.3]. In our case, these are the objects $S^n \mathcal{O}_U$ (the cofibers of the maps of J are trivial in the derived category). Because S is noetherian, the presheaf colimit of a direct system of sheaves coincides with the sheaf colimit [Har77, Exercise II.1.11]. It follows from this that $\mathrm{Ch}(\mathcal{O}\text{-Mod})(S^n \mathcal{O}_U, -)$ commutes with direct colimits. The results of [Hov98, Section 7.4] then show that $S^n \mathcal{O}_U$ is small (in the triangulated sense) in the derived category. $\qquad\square$

The derived category of $\mathcal{O}$-modules is known to be a symmetric monoidal triangulated category even without the finite hereditary global dimension assumption [Lip98, Section 2.5]. This might indicate that there is some replacement for the flat model structure that works more generally, or it might indicate that model categories are simply not adequate to cope with the general case.

To show that the derived category is in fact a unital algebraic stable homotopy category, we would need to know that the generators $\mathcal{O}_U$ are strongly dualizable. This would mean we would need to show that the natural map

$$R \operatorname{Hom}(\mathcal{O}_U, \mathcal{O}) \otimes \mathcal{O}_V \to R \operatorname{Hom}(\mathcal{O}_U, \mathcal{O}_V)$$

is a quasi-isomorphism. (We don't need the derived tensor product since $\mathcal{O}_V$ is cofibrant, using Corollary 5.7). Unfortunately, this is false in the simplest nontrivial example. Indeed, consider sheaves of abelian groups on the Sierpinski space S. Recall S has only two points, exactly one of which is open. Sheaves on S coincide with presheaves, which in turn coincide with maps $A \to B$ of abelian groups. In this case, $\mathcal{O}$ is the identity map $\mathbb{Z} \to \mathbb{Z}$, and, taking U to be the open point, $\mathcal{O}_U$ is the map $0 \to \mathbb{Z}$. One can then calculate to find $R \operatorname{Hom}(\mathcal{O}_U, \mathcal{O}) = R \operatorname{Hom}(\mathcal{O}_U, \mathcal{O}_U)$,

but this equality is destroyed on tensoring the left hand side with $\mathcal{O}_U$. A similar counterexample works if we think of S as the underlying space of $\operatorname{Spec} \mathbb{Z}_{(p)}$.

There is an additional condition that a symmetric monoidal model category might satisfy, called the *monoid axiom* [SS97]. This axiom guarantees that the monoids in a symmetric monoidal model category, and the modules over a given monoid, themselves form model categories. The monoid axiom asserts that every map in K-cof is a weak equivalence, where K is the class consisting of all maps $f \otimes X$, where f is a trivial cofibration and X is an arbitrary object.

Theorem 5.9. *Suppose $(S, \mathcal{O})$ is a ringed space with finite hereditary global dimension. The the flat model structure on $\operatorname{Ch}(\mathcal{O}\text{-}Mod)$ satisfies the monoid axiom.*

Proof. Suppose f is a flat trivial cofibration, and X is an arbitrary object. Then $0 \to X$ is a monomorphism, so applying Theorem 5.6 shows that $f \otimes X$ is an injective quasi-isomorphism. Corollary 1.7 completes the proof.   □

The following corollary follows immediately from Theorem 5.9 and [SS97].

Corollary 5.10. *Suppose $(S, \mathcal{O})$ is a ringed space with finite hereditary global dimension. Then:*

(a) *The category of monoids in $\operatorname{Ch}(\mathcal{O}\text{-}Mod)$ is a cofibrantly generated model category, where a map of monoids is a weak equivalence or a fibration if and only if it is so in the flat model structure on $\operatorname{Ch}(\mathcal{O}\text{-}Mod)$.*

(b) *Given a monoid R in $\operatorname{Ch}(\mathcal{O}\text{-}Mod)$, the category of R-modules, R-Mod, is a cofibrantly generated proper model category, where a map of modules is a weak equivalence or a fibration if and only if it is so in the flat model structure on $\operatorname{Ch}(\mathcal{O}\text{-}Mod)$.*

(c) *If R is a commutative monoid, then R-Mod is a symmetric monoidal model category satisfying the monoid axiom. Furthermore, the category of algebras over R is a cofibrantly generated model category, where a map of algebras is a weak equivalence or fibration if and only if it is so in the flat model structure on $\operatorname{Ch}(\mathcal{O}\text{-}Mod)$.*

The category of monoids will certainly be right proper, but we do not know if the category of monoids is left proper.

A map of monoids $R \to R'$ will induce the usual induction and restriction adjunction R-Mod $\to R'$-Mod. This adjunction will be a Quillen adjunction, but we would like something more.

Proposition 5.11. *Suppose $(S, \mathcal{O})$ is a ringed space with finite hereditary global dimension, and $R \to R'$ is a weak equivalence of monoids in $\operatorname{Ch}(\mathcal{O}\text{-}Mod)$. Then the induction and restriction adjunction R-Mod $\to R'$-Mod is a Quillen equivalence.*

Proof. It suffices to show that, if N is a cofibrant R-module, then $- \otimes_R N$ preserves weak equivalences, by [SS97]. The proof of this is very similar to the proof of the corresponding fact in Corollary 5.7, so we leave it to the reader.   □

We now investigate the functoriality of the flat model structure. Suppose we have a map of ringed spaces $f \colon (S, \mathcal{O}_S) \to (T, \mathcal{O}_T)$. Recall that this is a continuous map $f \colon S \to T$ together with a map of sheaves of rings $\mathcal{O}_T \to f_* \mathcal{O}_S$. Here, for any sheaf X on S, $f_*(X)$ is the sheaf on T defined by $f_*(X)(U) = X(f^{-1}(U))$. If X is an $\mathcal{O}_S$-module, then $f_* X$ is an $f_* \mathcal{O}_S$-module, and so an $\mathcal{O}_T$-module by restriction. The functor $f_* \colon \mathcal{O}_S$-Mod $\to \mathcal{O}_T$-Mod has a left adjoint f^*. To define this, recall

that if Y is a sheaf on T, $f^{-1}Y$ is the sheaf on S associated to the presheaf that takes U to $\operatorname{colim}_{V \supseteq f(U)} Y(V)$. The functor f^{-1} is left adjoint to f_* on the category of sheaves of abelian groups, so in particular we have a map $f^{-1}\mathcal{O}_T \to \mathcal{O}_S$. Given an $\mathcal{O}_T$-module Y, we define $f^*Y = \mathcal{O}_S \otimes_{f^{-1}\mathcal{O}_T} f^{-1}Y$. It is well-known that f^* is left adjoint to f_* and is symmetric monoidal [Gro60, Section 0.4.3].

One can verify using adjointness that, if U is an open subset of T, then $f^*\mathcal{O}_U = \mathcal{O}_{f^{-1}U}$. Hence we have the following proposition.

Proposition 5.12. *Suppose* $f \colon (S, \mathcal{O}_S) \to (T, \mathcal{O}_T)$ *is a map of ringed spaces with finite hereditary global dimension. Then* f^* *is a left Quillen functor with respect to the flat model structures.*

In particular, this shows that the total left derived functor of f^* exists and is left adjoint to the total right derived functor of f_*. It is proved in [Lip98, Section 2.7] that the total left derived functor of f^* exists without the finite hereditary global dimension hypotheses. It is disconcerting that we are unable to reproduce this result using model categories.

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